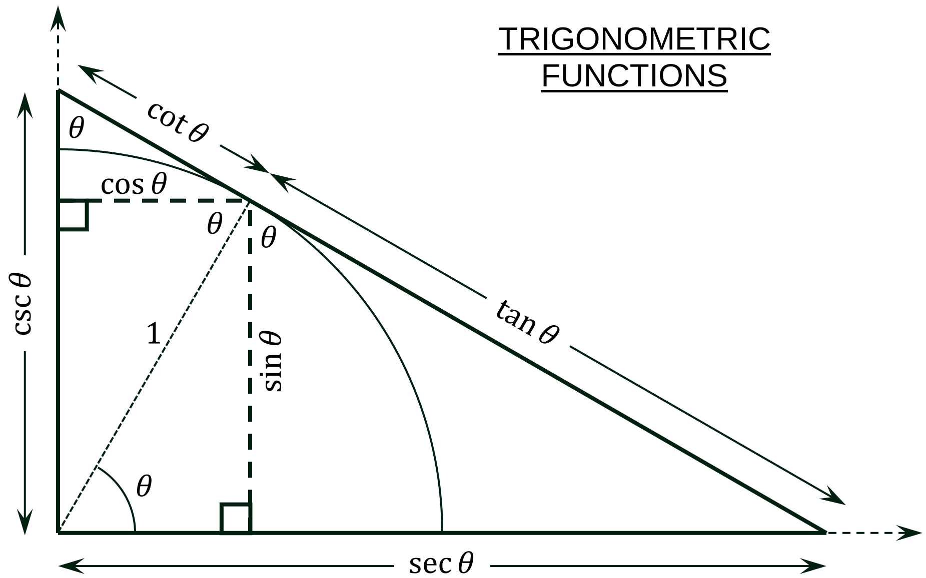
TRIGONOMETRIC FUNCTIONS



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

RECIPROCAL IDENTITIES

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

RATIO IDENTITIES

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
$\tan(\theta) = \frac{\sec(\theta)}{\csc(\theta)}$	$\cot(\theta) = \frac{\csc(\theta)}{\sec(\theta)}$

PYTHAGOREAN IDENTITIES

$$\cos(\theta) = \frac{A}{H}$$

$$\sin(\theta) = \frac{O}{H}$$

$$\sin^2(\theta) + \sin^2(\theta) = 1$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

SYMMETRY IDENTITIES

The graph of $\sin(\theta)$ is symmetric about the origin, therefore…

$$\sin(-\theta) = -\sin(\theta)$$

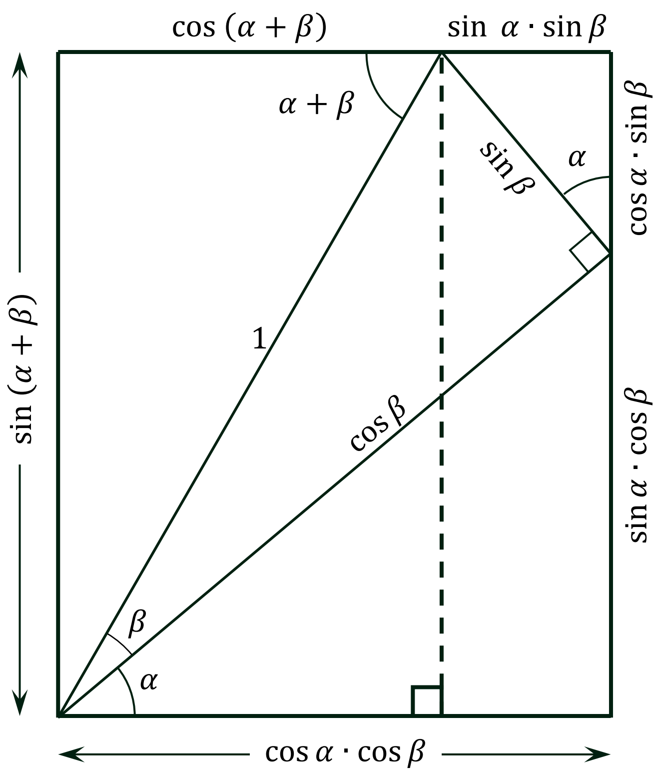
The graph of $\cos(\theta)$ is symmetric about the y-axis, therefore…

$$\cos(-\theta) = \cos(\theta)$$

The graph of $\tan(\theta)$ is symmetric about the origin, therefore…

$$\tan(-\theta) = -\tan(\theta)$$

SIN and COS of SUMS (and DIFFERENCES)



$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

$$\sin(\alpha-\beta) = \sin\big(\alpha+(-\beta)\big)$$
$$= \sin\alpha\cdot\cos(-\beta) + \cos\alpha\cdot\sin(-\beta)$$

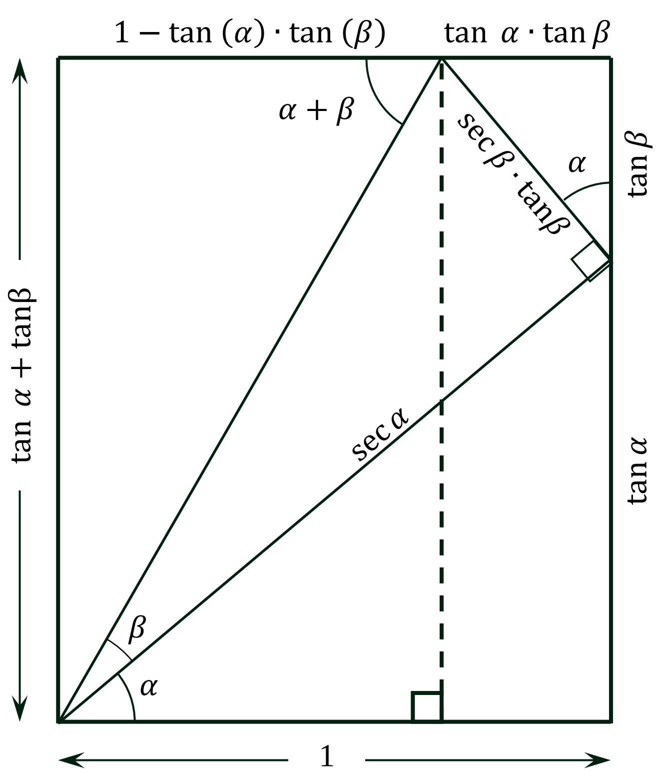
$$\sin(\alpha-\beta) = \sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

$$\cos(\alpha-\beta) = \cos\big(\alpha+(-\beta)\big)$$
$$= \cos\alpha\cdot\cos(-\beta) - \sin\alpha\cdot\sin(-\beta)$$

$$\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$$

TAN of SUMS (and DIFFERENCES)



$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$\tan(-\beta) = -\tan\beta$$

$$\tan(\alpha-\beta) = \tan\big(\alpha+(-\beta)\big)$$
$$= \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\cdot\tan(-\beta)}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

DOUBLE ANGLE FORMULAE

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$\sin(2\theta) = \sin(\theta + \theta) = \sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta$$

$$\boxed{\sin(2\theta) = 2 \cdot \sin\theta \cdot \cos\theta}$$

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos\theta \cdot \cos\theta - \sin\theta \cdot \sin\theta$$

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$$

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta \quad \boxed{= 1 - 2 \cdot \sin^2\theta}$$

$$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta) \quad \boxed{= 2 \cdot \cos^2\theta - 1}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta \cdot \tan\theta}$$

$$\boxed{\tan(2\theta) = \frac{2 \cdot \tan\theta}{1 - \tan^2\theta}} \quad \cdot \left(\frac{\cos^2\theta}{\cos^2\theta}\right) =$$

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} \quad \boxed{= \frac{2 \cdot \sin\theta \cdot \cos\theta}{\cos^2\theta - \sin^2\theta}}$$

HALF ANGLE FORMULAE

$$\cos(2\theta) = 1 - 2\cdot\sin^2\theta \qquad \cos(2\theta) = 2\cdot\cos^2\theta - 1$$

$$2\cdot\sin^2\theta = 1 - \cos(2\theta) \qquad 2\cdot\cos^2\theta = 1 + \cos(2\theta)$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}} \qquad \cos\theta = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\frac{\emptyset}{2} = \theta$$

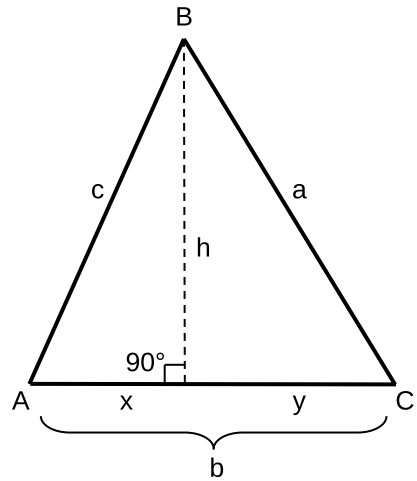
$$\boxed{\sin\frac{\emptyset}{2} = \pm\sqrt{\frac{1 - \cos(\emptyset)}{2}}} \qquad \boxed{\cos\frac{\emptyset}{2} = \pm\sqrt{\frac{1 + \cos(\emptyset)}{2}}}$$

$$\tan\frac{\emptyset}{2} = \frac{\sin\frac{\emptyset}{2}}{\cos\frac{\emptyset}{2}} = \frac{\pm\sqrt{\frac{1 - \cos(\emptyset)}{2}}}{\pm\sqrt{\frac{1 + \cos(\emptyset)}{2}}}$$

$$\tan\frac{\emptyset}{2} = \frac{\sqrt{1 - \cos(\emptyset)}}{\sqrt{1 + \cos(\emptyset)}}\cdot\frac{\sqrt{1 + \cos(\emptyset)}}{\sqrt{1 + \cos(\emptyset)}} = \frac{\sqrt{1 - \cos^2(\emptyset)}}{1 + \cos(\emptyset)} \boxed{= \frac{\sin(\emptyset)}{1 + \cos(\emptyset)}}$$

$$\tan\frac{\emptyset}{2} = \frac{\sqrt{1 - \cos(\emptyset)}}{\sqrt{1 + \cos(\emptyset)}}\cdot\frac{\sqrt{1 - \cos(\emptyset)}}{\sqrt{1 - \cos(\emptyset)}} = \frac{1 - \cos(\emptyset)}{\sqrt{1 - \cos^2(\emptyset)}} \boxed{= \frac{1 - \cos(\emptyset)}{\sin(\emptyset)}}$$

$$\boxed{\tan\frac{\emptyset}{2} = \pm\sqrt{\frac{1 - \cos(\emptyset)}{1 + \cos(\emptyset)}}}$$



LAW of SINES DERIVATION

Equation 1 $h = c \cdot \sin A$

Equation 2 $h = a \cdot \sin C$

Equate the right sides of equations 1 and 2 and rearrange…

Equation 3 $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

Note: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ and $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ are derived similarly.

Law of Sines… $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$ or $\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$

LAW of COSINES DERIVATION

Equation 4 $h^2 = a^2 - y^2 = a^2 - (b - x)^2$

Equation 5 $h^2 = c^2 - x^2$

Equate the right sides of equations 4 and 5 and rearrange…

$a^2 - (b - x)^2 = c^2 - x^2$

Equation 6 $a^2 = b^2 + c^2 - 2bx$

Equation 7 $x = c \cdot \cos A$

Substitute for x from equation 7 into equation 6…

Equation 8 $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Note: equations for b^2 and c^2 are derived similarly.

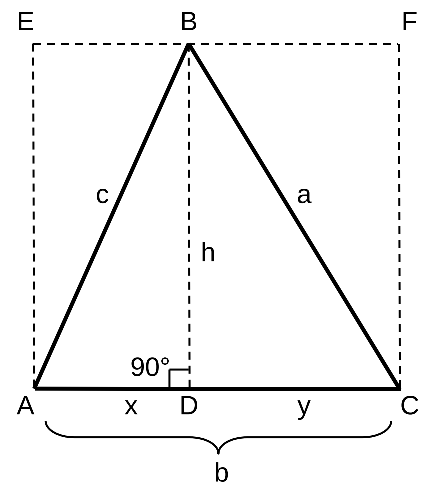
Law of Cosines…

alternate versions

$\boxed{\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}}$ or $\boxed{a^2 = b^2 + c^2 - 2bc \cdot \cos A}$

$\boxed{\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}}$ or $\boxed{b^2 = a^2 + c^2 - 2ac \cdot \cos B}$

$\boxed{\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}}$ or $\boxed{c^2 = a^2 + b^2 - 2ab \cdot \cos C}$



Area ABD = Area ABE = $\frac{x \cdot h}{2}$

Area CBD = Area CBF = $\frac{y \cdot h}{2}$

Therefore…

Area of Triangle ABC…

$$Area = \frac{b \cdot h}{2}$$

$$h = a \cdot \sin C$$

Therefore…

Area of Triangle ABC…

$$Area = \frac{a \cdot b \cdot \sin C}{2}$$

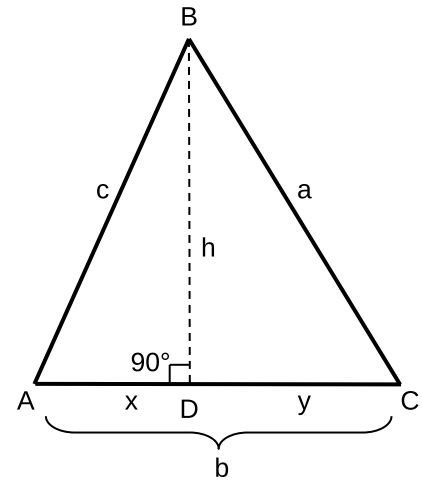
$$b = \frac{a \cdot \sin B}{\sin A}$$

Therefore…

Area of Triangle ABC…

$$Area = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot \sin A}$$

HERON'S FORMULA DERIVATION



Equation 1 $Area = \frac{b \cdot h}{2}$

From triangle ABD… $x^2 + h^2 = c^2$

Equation 2 $x^2 = c^2 - h^2$

Equation 3 $x = \sqrt{c^2 - h^2}$

From triangle CBD… $(b - x)^2 + h^2 = a^2$

$$(b - x)^2 = a^2 - h^2$$

Equation 4 $b^2 - 2bx + x^2 = a^2 - h^2$

Substitute x and x^2 from Equations 3 and 2 into Equation 4 …

$$b^2 - 2b\sqrt{c^2 - h^2} + (c^2 - h^2) = a^2 - h^2$$

$$b^2 + c^2 - a^2 = 2b\sqrt{c^2 - h^2}$$

Square both sides … $(b^2 + c^2 - a^2)^2 = 4b^2(c^2 - h^2)$

$$\frac{(b^2+c^2-a^2)^2}{4b^2} = c^2 - h^2$$

$$h^2 = c^2 - \frac{(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{4b^2c^2-(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{(2bc)^2-(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{[2bc+(b^2+c^2-a^2)]\cdot[2bc-(b^2+c^2-a^2)]}{4b^2}$$

$$h^2 = \frac{[2bc+b^2+c^2-a^2]\cdot[2bc-b^2-c^2+a^2]}{4b^2}$$

$$h^2 = \frac{[(b^2+2bc+c^2)-a^2]\cdot[a^2-(b^2-2bc+c^2)]}{4b^2}$$

$$h^2 = \frac{\left[(b+c)^2-a^2\right]\cdot\left[a^2-(b-c)^2\right]}{4b^2}$$

$$h^2 = \frac{[(b+c)+a]\cdot[(b+c)-a]\cdot[a+(b-c)]\cdot[a-(b-c)]}{4b^2}$$

$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}{4b^2}$$

Since $P = a + b + c$ …

$$h^2 = \frac{P(P-2a)(P-2b)(P-2c)}{4b^2}$$

<u>Equation 5</u>

$$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

Substitute h from Equation 5 into Equation 1 …

$$Area = \frac{1}{2} b \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

$$Area = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)}$$

$$Area = \sqrt{\frac{1}{16}P(P-2a)(P-2b)(P-2c)}$$

$$Area = \sqrt{\left(\frac{P}{2}\right)\left(\frac{P-2a}{2}\right)\left(\frac{P-2b}{2}\right)\left(\frac{P-2c}{2}\right)}$$

$$Area = \sqrt{\frac{P}{2}\left(\frac{P}{2}-a\right)\left(\frac{P}{2}-b\right)\left(\frac{P}{2}-c\right)}$$

Thus...

Given the three sides of a triangle (a, b, and c) …

The area of the triangle is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where the semi perimeter is:

$$s = P/2 = (a+b+c)/2$$